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A Canonical Approach to Self-Duality of Dirichlet 3-Brane

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Abstract

The self-duality of Dirichlet 3-brane action under the $SL(2, R)$ duality transformation of type IIB superstring theory is shown in the Hamiltonian form of the path integral for the partition function by performing the direct integration with respect to the boundary gauge field. Through the integration in the phase space the canonical momentum conjugate to the boundary gauge field can be effectively replaced by the dual gauge field.

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Recent developments in the type II superstring theory have shown that p-brane solutions and strong-weak coupling dualities play an important role in extracting the non-perturbative properties of the theory [1, 2, 3]. The supersymmetric p-brane solitonic solutions [3, 4] carrying R-R charges are described by open strings with Dirichlet boundary conditions in the transverse directions and Neumann boundary conditions in the $p+1$ world-volume directions [5, 6, 7, 8]. Through the investigations of the D-p-brane by the conformal field theory approach the soliton dynamics has been elucidated [9, 10, 11, 12, 13]. The action of the D-p-brane for $0 \leq p \leq 3$ has been constructed to have a kinetic term of the Dirac-Born-Infeld (DBI) type [6] showing the coupling to the NS-NS background fields and a Wess-Zumino term indicating the coupling to the R-R background fields as well as the NS-NS 2-form gauge field [14, 15, 16, 17, 18, 19]. The modified gauge-invariant field strength provided by both the boundary gauge field and the NS-NS 2-form gauge field exists in the two terms. The boundary gauge field is the open-string boundary condensate of the electromagnetic field. From the integration over the boundary gauge field for the D-string and D-2-brane actions the fundamental string action with tension given by Schwarz's formula [20] and the bosonic part of the eleven-dimensional supermembrane action [21] have been derived respectively, by the Lagrangian formulation introducing the auxiliary fields [15] or by the canonical Hamiltonian approach [16]. The D-string case provides support for the predictions of $SL(2, Z)$ duality of the type IIB theory. From a different point of view through the introduction of the Lagrange multiplier field to make the semiclassical duality transformation for the boundary gauge field these two actions have been reproduced and further in the D-3-brane case the transformed action has turned out to be the original form in terms of the dualized boundary gauge and background fields [17]. It indicates that the D-3-brane action is invariant under the $SL(2, R)$ duality transformations of the background fields of type IIB theory combined with world-volume vector duality in the four dimensions. This duality invariance has been found in a similar way as the four-dimensional non-linear electrodynamics [22].

Here we will explore the D-3-brane in the canonical Hamiltonian approach. Without resort to the Lagrange multiplier field we will try to integrate the canonical boundary gauge field directly in the path integral of the Hamiltonian form. The obtained effective action will get back to the original one in terms of dualized fields. We wil see how the dual variable for the boundary gauge field appears in this direct integration.

We start to write the D-3-brane action of type IIB theory

$$\begin{aligned} S_D &= \int d^4x[e^{-\phi}\sqrt{-\det(G_{\alpha\beta}+\mathcal{F}_{\alpha\beta})} \\ &\quad +\frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}(\frac{1}{3}C_{\alpha\beta\gamma\delta}+2C_{\alpha\beta}\mathcal{F}_{\gamma\delta}+C\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta})], \end{aligned} \tag{1}$$

which is the source term of the D-3-brane when combined with the type IIB effective action. This closed-string effective action was presented under some restriction on the R-R 4-form gauge field [23, 24] and was shown to be invariant under the $SL(2, R)$ duality transformation of the type IIB theory [1, 2, 20, 23, 25]. The kinetic term is described

by a dilaton ϕ, a metric tensor $G_{\mu\nu}$ and an antisymmetric tensor $B^{(1)}_{\mu\nu}$ in the NS sector. The embedding metric is given by $G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} (\alpha, \beta = 0, \ldots, 3)$ where X^μ is the coordinates of the ten-dimensional target space of type IIB theory. From the boundary gauge field A_α to be mixed with the closed-string antisymmetric 2-form gauge field the modified gauge-invariant field strength is defined as $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} - B^{(1)}_{\alpha\beta}$ with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, B^{(1)}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu B^{(1)}_{\mu\nu}$. The fields such as $C, C_{\alpha\beta} \equiv B^{(2)}_{\alpha\beta}, C_{\alpha\beta\gamma\delta}$ are also the induced background R-R fields of type IIB theory. For simplicity we consider the case where the spacetime metric is Minkowskian. The DBI Lagrangian is expressed as

$$\sqrt{-\det(G_{\alpha\beta} + \mathcal{F}_{\alpha\beta})} = \sqrt{1 - \boldsymbol{E}^2 + \boldsymbol{B}^2 - (\boldsymbol{B} \cdot \boldsymbol{E})^2}, \tag{2}$$

where $E_i (i = 1, 2, 3)$ stands for $\mathcal{F}_{0i}$ and B_i for $\frac{1}{2}\epsilon_{ijk}\mathcal{F}_{jk}$. The canonical momenta conjugate to the world-volume gauge fields A_α are given by

$$\pi_0 = 0, \boldsymbol{\pi} = -\frac{\boldsymbol{E} + (\boldsymbol{B} \cdot \boldsymbol{E})\boldsymbol{B}}{e^\phi \sqrt{D}} + \boldsymbol{C} + C\boldsymbol{B}, \tag{3}$$

where $C_i = \frac{1}{2}\epsilon_{ijk}C_{jk}$ and $\sqrt{D}$ denotes the expression (2). We pass to the Hamiltonian

$$H = p_i E_i - e^{-\phi}\sqrt{D} + (\partial_i A_0 + B^{(1)}_{0i})\pi_i - (C_{0123} + C_{0i}B_i), \tag{4}$$

where

$$p_i = \pi_i - C_i - CB_i. \tag{5}$$

In the non-linear equation (3) with (2), (5) we will derive a solution for $\boldsymbol{E}$. First by using (3) and (5) we get a relation

$$\boldsymbol{E} = -\frac{e^\phi \sqrt{D}}{1 + \boldsymbol{B}^2}\boldsymbol{Q}, \tag{6}$$

where

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 1 + B_2^2 + B_3^2 & -B_1 B_2 & -B_1 B_3 \\ -B_2 B_1 & 1 + B_3^2 + B_1^2 & -B_2 B_3 \\ -B_3 B_1 & -B_3 B_2 & 1 + B_1^2 + B_2^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}. \tag{7}$$

The next step is to substitute the relation into (2), which yields consistently

$$\sqrt{D} = \frac{(1 + \boldsymbol{B}^2)^{3/2}}{e^\phi \sqrt{M}} \tag{8}$$

with $M = (1 + \boldsymbol{B}^2)^2 e^{-2\phi} + \boldsymbol{Q}^2 + (\boldsymbol{B} \cdot \boldsymbol{Q})^2$. Combining (6) and (8) we obtain a solution $\boldsymbol{E} = -\sqrt{(1 + \boldsymbol{B}^2)/M}\boldsymbol{Q}$. Owing to the solution the combination $p_i E_i - e^{-\phi}\sqrt{D}$ in (4) turns out to be $-\sqrt{(1 + \boldsymbol{B}^2)}[e^{-2\phi}(1 + \boldsymbol{B}^2) + \boldsymbol{Q} \cdot \boldsymbol{p}]/\sqrt{M}$. As M can be shown to be equal to $(1 + \boldsymbol{B}^2)[e^{-2\phi}(1 + \boldsymbol{B}^2) + \boldsymbol{Q} \cdot \boldsymbol{p}]$, we have

$$H = -\sqrt{e^{-2\phi}(1 + \boldsymbol{B}^2) + \boldsymbol{Q} \cdot \boldsymbol{p}} + (\partial_i A_0 + B^{(1)}_{0i})\pi_i - (C_{0123} + C_{0i}B_i). \tag{9}$$

Now we are ready to write the partition function by using the Hamiltonian form of the path integral as

$$\begin{aligned} Z &= \int \prod_{i=1}^{3}[d\pi_i] \prod_{\alpha=0}^{3}[dA_\alpha] \exp[i\int d^4x(\pi_i \dot{A}_i - H(A,\pi))] \\ &= \int [d(\pi, A)] \exp[i\int d^4x\{\partial_i\pi_i A_0 + \pi_i(\partial_0 A_i - B_{0i}^{(1)}) + \sqrt{N} + C_{0123} + C_{0i}B_i\}], \end{aligned} \quad (10)$$

where the integration over π_0 is canceled by the gauge group volume and $N = (1 + \boldsymbol{B}^2)(e^{-2\phi} + \boldsymbol{p}^2) - (\boldsymbol{p}\cdot\boldsymbol{B})^2$. The integration over A_0 brings a δ-function $\delta(\partial_i\pi_i)$ down to the path integral, which allows us to write $\pi_i = \epsilon_{ijk}\partial_j D_k$ with a vector field D_i. In the integration over A_i the magnetic fields B_i^0 defined by $B_i^0 = \epsilon_{ijk}\partial_j A_k$ are not independent fields because of $\partial_i B_i^0 = 0$. We make change of integration variables from A_i to B_i^0 and treat the magnetic fields B_i^0 as the independent variables but with a constraint $\partial_i B_i^0 = 0$, whose δ-function is expressed by introducing an auxiliary field D_0 as

$$\prod_x \delta(\partial_i B_i^0(x)) = \int [dD_0] \exp(-i\int d^4x D_0 \partial_i B_i^0). \quad (11)$$

Then the partition function is described by

$$Z = \int \prod_{\alpha=0}^{3}[dD_\alpha] \prod_{i=1}^{3}[dB_i^0] \exp[i\int d^4x\{\sqrt{N} - B_i(D_{0i} - C_{0i}) + \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}(\frac{1}{3}C_{\alpha\beta\gamma\delta} - 2D_{\alpha\beta}B_{\gamma\delta}^{(1)})\}] \quad (12)$$

with $D_{\alpha\beta} = \partial_\alpha D_\beta - \partial_\beta D_\alpha$. Through a shift the integration over B_i^0 becomes that over B_i.

Here we make the integration over B_i by using the saddle point approximation. It is noted that

$$N = (1 + \boldsymbol{B}^2)e^{-2\phi} + (\bar{\boldsymbol{p}} - C\boldsymbol{B})^2 + (\bar{\boldsymbol{p}} \times \boldsymbol{B})^2, \quad (13)$$

where $\bar{p}_i = \frac{1}{2}\epsilon_{ijk}(D_{jk} - C_{jk})$. The variation with respect to B_i yields the saddle point equation

$$\begin{pmatrix} \Lambda + \bar{p}_2^2 + \bar{p}_3^2 & -\bar{p}_1\bar{p}_2 & -\bar{p}_1\bar{p}_3 \\ -\bar{p}_2\bar{p}_1 & \Lambda + \bar{p}_3^2 + \bar{p}_1^2 & -\bar{p}_2\bar{p}_3 \\ -\bar{p}_3\bar{p}_1 & -\bar{p}_3\bar{p}_2 & \Lambda + \bar{p}_1^2 + \bar{p}_2^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \sqrt{N}d_{01} + C\bar{p}_1 \\ \sqrt{N}d_{02} + C\bar{p}_2 \\ \sqrt{N}d_{03} + C\bar{p}_3 \end{pmatrix}, \quad (14)$$

where $\Lambda = e^{-2\phi} + C^2, d_{0i} = D_{0i} - C_{0i}$. We use the inversion of the matrix

$$\frac{1}{\Lambda(\Lambda + \bar{\boldsymbol{p}}^2)} \begin{pmatrix} \Lambda + \bar{p}_1^2 & \bar{p}_1\bar{p}_2 & \bar{p}_1\bar{p}_3 \\ \bar{p}_2\bar{p}_1 & \Lambda + \bar{p}_2^2 & \bar{p}_2\bar{p}_3 \\ \bar{p}_3\bar{p}_1 & \bar{p}_3\bar{p}_2 & \Lambda + \bar{p}_3^2 \end{pmatrix} \quad (15)$$

to have

$$\boldsymbol{B} = \frac{1}{\Lambda(\Lambda + \bar{\boldsymbol{p}}^2)}[\sqrt{N}\{\Lambda\boldsymbol{d} + (\bar{\boldsymbol{p}}\cdot\boldsymbol{d})\bar{\boldsymbol{p}}\} + (\Lambda + \bar{\boldsymbol{p}}^2)C\bar{\boldsymbol{p}}] \quad (16)$$

with $d_i = d_{0i}$. Substituting this non-linear relation back into (13) to observe the cancellation of the $\sqrt{N}$ term, we can determine N consistently as

$$N = \frac{e^{-2\phi}(\Lambda + \bar{\boldsymbol{p}}^2)^3}{\Lambda(\Lambda + \bar{\boldsymbol{p}}^2)^2 - \Lambda^2 \boldsymbol{d}^2 - (\bar{\boldsymbol{p}} \cdot \boldsymbol{d})^2(2\Lambda + \bar{\boldsymbol{p}}^2) - \Lambda(\bar{\boldsymbol{p}} \times \boldsymbol{d})^2}. \tag{17}$$

In this self-consistent way we find the explicit solution (16) accompanied with (17) for the non-linear saddle point equation. Plunging the complicated solution into the action in (12), we have

$$\begin{aligned} \int d^4x \quad [& \quad \frac{\Lambda(\Lambda + \bar{\boldsymbol{p}}^2) - \Lambda \boldsymbol{d}^2 - (\bar{\boldsymbol{p}} \cdot \boldsymbol{d})^2}{\Lambda(\Lambda + \bar{\boldsymbol{p}}^2)} \sqrt{N} \\ & + \quad \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}(\frac{1}{3}C_{\alpha\beta\gamma\delta} - 2B^{(1)}_{\alpha\beta}D_{\gamma\delta} - \frac{C}{\Lambda}\mathcal{D}_{\alpha\beta}\mathcal{D}_{\gamma\delta})], \end{aligned} \tag{18}$$

where $\mathcal{D}_{\alpha\beta} = D_{\alpha\beta} - C_{\alpha\beta}$. Further N can be so rewritten as

$$N = \frac{e^{-2\phi}(\Lambda + \bar{\boldsymbol{p}}^2)^2}{\Lambda(\Lambda + \bar{\boldsymbol{p}}^2) - \Lambda \boldsymbol{d}^2 - (\bar{\boldsymbol{p}} \cdot \boldsymbol{d})^2} \tag{19}$$

that for the kinetic part we derive a compact expression $e^{-\phi}\sqrt{-\det(\sqrt{\Lambda}\eta_{\alpha\beta} + \mathcal{D}_{\alpha\beta})/\Lambda}$, which is just of the DBI type. This result producing the Lorentz invariant expression can be generalized to the general metric $G_{\alpha\beta}$ as

$$\begin{aligned} Z \quad &= \quad \int \prod_{\alpha=0}^{3}[dD_\alpha] \exp[i \int d^4x \{ e^{-\tilde{\phi}}\sqrt{-\det(\tilde{G}_{\alpha\beta} + \mathcal{D}_{\alpha\beta})} \\ & + \quad \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}(\frac{1}{3}C_{\alpha\beta\gamma\delta} + 2\tilde{C}_{\alpha\beta}\mathcal{D}_{\gamma\delta} + \tilde{C}\mathcal{D}_{\alpha\beta}\mathcal{D}_{\gamma\delta})\}], \end{aligned} \tag{20}$$

where $\mathcal{D}_{\alpha\beta} = D_{\alpha\beta} - \tilde{B}^{(1)}_{\alpha\beta}$ and

$$\begin{aligned} \tilde{B}^{(1)}_{\alpha\beta} = C_{\alpha\beta}, \qquad & \tilde{C}_{\alpha\beta} = -B^{(1)}_{\alpha\beta}, \\ e^{-\tilde{\phi}} = \frac{e^{-\phi}}{\Lambda} = \frac{1}{e^{-\phi} + e^{\phi}C^2}, \quad & \tilde{C} = -\frac{C}{\Lambda} = -\frac{Ce^{\phi}}{e^{-\phi} + e^{\phi}C^2}, \\ \tilde{G}_{\alpha\beta} = \sqrt{\Lambda}G_{\alpha\beta} = e^{(\tilde{\phi}-\phi)/2}G_{\alpha\beta}. & \end{aligned} \tag{21}$$

The obtained relations in (21) show the basic inversion $\lambda \to -1/\lambda$ in the $SL(2,R)$ duality transformation , which is specified by $p = 0, q = 1, r = -1, s = 0$ for

$$\begin{aligned} g_{\alpha\beta} \to g_{\alpha\beta}, \quad & \lambda \to (p\lambda + q)/(r\lambda + s), \\ B^{(1)}_{\alpha\beta} \to sB^{(1)} - rC_{\alpha\beta}, \quad & C_{\alpha\beta} \to pC_{\alpha\beta} - qB^{(1)}_{\alpha\beta} \end{aligned} \tag{22}$$

with $g_{\alpha\beta} = e^{-\phi/2} G_{\alpha\beta}, \lambda = C + ie^{-\phi}$. Although D_i and D_0 are separately induced, the final result shows that they are combined into a four-dimensional dual vector. This dualized action was presented in the Lagrangian formulation for the duality transformation by adding the Lagrange multiplier term $\frac{1}{2}\Lambda_{\alpha\beta}(\mathcal{F}_{\alpha\beta} - 2\partial_\alpha A_\beta + B^{(1)}_{\alpha\beta})$ to the D-3-brane action to symplify the integration over the boundary gauge field and further integrating over $\mathcal{F}_{\alpha\beta}$ in a special Lorentz frame where $\mathcal{F}_{\alpha\beta}$ is block-diagonal with eigenvalues f_1, f_2 by means of the saddle point approximation for f_1, f_2 [17]. In our Hamiltonian approach without choosing such a special Lorentz frame we have performed the $\mathcal{F}_{ij}$ integration in the arbitrary Lorentz frame to derive the dualized action. Together with the simple invariance under the shift of C we have given the alternative way to show the D-3-brane action is invariant under the full $SL(2, R)$ duality transformation.

Here we return to the path integral (10) in the phase space. Alternatively if the integration with respect to the canonical momentum π_i is evaluated also by the saddle point value, which is in the similar way given by

$$\boldsymbol{\pi} = -\frac{\sqrt{N'}}{1 + \boldsymbol{B}^2}(\boldsymbol{E} + (\boldsymbol{B} \cdot \boldsymbol{E})\boldsymbol{B}) + \boldsymbol{C} + C\boldsymbol{B} \tag{23}$$

with

$$N' = \frac{e^{-2\phi}(1 + \boldsymbol{B}^2)^2}{1 - \boldsymbol{E}^2 + \boldsymbol{B}^2 - (\boldsymbol{B} \cdot \boldsymbol{E})^2}, \tag{24}$$

then we go back to the starting classical D-3-brane action itself (1) as expected. It is observed that these expressions (23), (24) show similar behaviors to (16), (19), respectively.

In the Hamiltonian approach to the D-3-brane we have seen that the constraint $\partial_i \pi_i = 0$ provided from the A_0 integration brings about the dual variable D_i as $\pi_i = \epsilon_{ijk}\partial_j D_k$, while for the D-string and D-2-brane cases each canonical momentum is described by $\pi = q_1$ and $\pi_i = \epsilon_{ij}\partial_j y$ where q_1 is a charge of the dyonic string solution in type IIB theory and the scalar function y becomes an additional target space coordinate in the eleven-dimensional target space of the fundamental 2-brane [16]. In transforming the variables from A_i to F_{ij} an extra dual variable D_0 has appeared, which is a conspicuous behavior in the D-3-brane case in contrast to the D-string and D-2-brane cases. Moreover when integrating with respect to the magnetic field F_{ij} we have met the non-linear saddle point equation, which is compared with the linear ones in the D-string and D-2-brane cases. In our Hamiltonian approach the canonical momentum π_α has been replaced by the dual variable D_α and this change of variable has been generated as a result of the integration over A_α.

In conclusion, starting from the D-3-brane action we have constructed the Hamiltonian by solving the non-linear equation in order to obtain the electric field of the boundary gauge field and integrated over the magnetic field by solving the non-linear saddle point equation in the path integral of the Hamiltonian form to derive the effective action, that is just the dualized action in terms of the dual gauge field and the basic $SL(2, R)$ transformed background fields of the type IIB theory. The essential ingredient in this

Hamiltonian approach is to look for the explicit solutions of these two non-linear equations. They have been solved analogously in a self-consistent manner. Although the derived solutions are highly involved, the resulting effective action is put in the compact form. In the transformation of the integration variables from the three space-components of the boundary gauge field to the magnetic fields we need a new field, which turns into a time-component of the dual gauge field and plays an important role to obtain the Lorentz-invariant expression of the dualized action. Our demonstration of the self-duality of the D-3-brane action is a direct approach in the sense that the extraneous structure as the Lagrange multiplier field does not need to be introduced from the starting point. We hope that our prescription sheds insight into the structure of duality transformation from the Hamiltonian point of view in the phase space of the theory.

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